# Efficient Utilization of Fourier Transform Linearity Property

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Many Fourier transform FT problems need long time solutions, so FT properties facilitate the solution of these problems and may be evaluated almost by inspection. It is necessary to study how these properties allow the easy solution of these problems with efficient utilization.

In the time domain, the signal of constant amplitude does not contain frequency components while the signal of variable amplitude does contain frequency components. Fourier transform (Fourier integrals) provides a means of analyzing and designing frequency selective filters for the separation of signals on the basis of their frequency contents, to obtain the continuous frequency spectrum of a given aperiodic signal x(t) such as energy signal and deterministic signal which has infinite periodic time $T_o$ , finite energy, and zero average power. Fourier transform is used for evaluating the frequency contents of the aperiodic signal x(t) and in a limiting sense for the periodic signal $x_p(t)$, Fig.1.

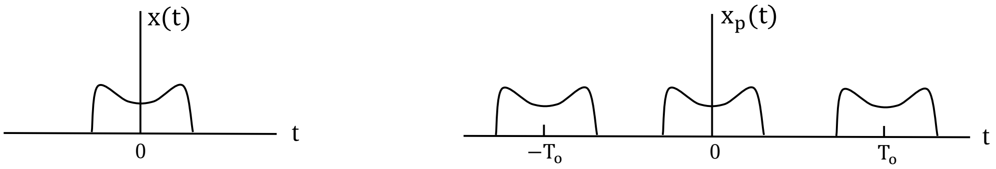



Fig.1. In the limit, as the period $T_o$ approaches to infinity, $x_p(t)$ approaches to x(t).

Since the aperiodic signal x(t) defines one cycle of the periodic signal $x_p(t)$, Fig.1, so x(t) is considered the generating function of $x_p(t)$ in the periodic time $T_o$ , and in the limit, as the periodic time $T_o$ approaches to infinity, Eq.(1), the periodic signal $x_p(t)$ approaches to the energy signal x(t), and x(t) yields

$$x(t) = \lim_{T_o \to \infty} x_p(t) \tag{1}$$

Fourier transform maps the energy signal x(t) from the time domain to its frequency domain X(f) in terms of the exponential functions and are related by the following Fourier transform integrals

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \tag{2}$$

, and

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df \tag{3}$$

, x(t) and X(f) constitutes a Fourier transform pair, and each of them can be obtained in terms of the other, Equations (2) and (3). The Fourier transform exists if the integral of Eq.(2), is absolutely integrable, that is the signal x(t) must energy signal has finite energy, according to the following Dirichlet`s conditions, Equations (4) and (5)

$$\int_{-\infty}^{\infty} |x(t)|\ dt < \infty \tag{4}$$

, and

$$\int_{-\infty}^{\infty} |x(t)|^2\ dt < \infty \tag{5}$$

A most convenient symbol of the Fourier transform pair and more frequently used, is

$$x(t) \rightleftharpoons X(f)$$

For a real valued energy signal x(t), the continuous amplitude spectrum |X(f)| is even function, and the continuous phase spectrum arg[X(f)] is an odd function, the Fourier transform pair yields.

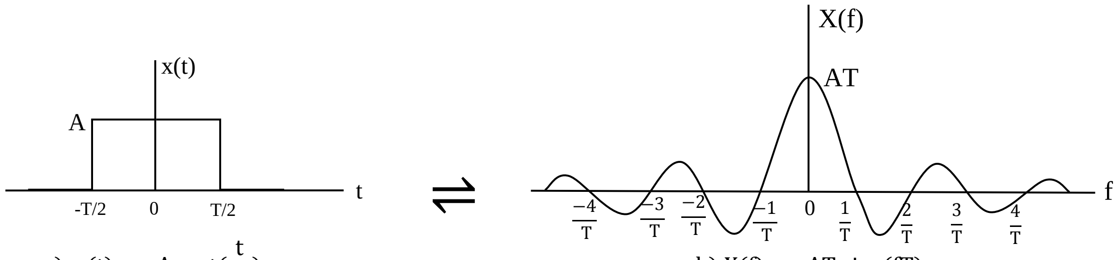
$$x(t) \rightleftharpoons |X(f)|\, e^{j\arg[X(f)]}$$

The Fourier transform X(f) is real value when the signal x(t) is an even function, imaginary value when the signal x(t) is an odd function, and complex value if the aperiodic signal x(t) is neither even nor odd function.

1. **For the Rectangular Pulse and its Sinc Spectra:** Consider x(t) an energy signal, is a rectangular pulse (gate function), centered at the origin, having amplitude A and duration T, is expressed by x(t) = A rect(t/T), its frequency spectrum X(f), Fig.2, is given by

$$X(f) = \int_{-T/2}^{T/2} A\, e^{-j2\pi ft}\, dt \;=\; AT\, \frac{\sin(\pi fT)}{\pi fT} = AT\, \mathrm{sinc}(fT)$$

, or equivalently $$A\, \mathrm{rect}(\frac{t}{T}) \rightleftharpoons AT\, \mathrm{sinc}(fT) \qquad (6)$$




a) $x(t) = A\, \mathrm{rect}(\frac{t}{T})$ b) $X(f) = AT\, \mathrm{sinc}(fT)$

Fig.2. a) Rectangular pulse, b) Sinc spectrum.

, such as the following Fourier transform pairs

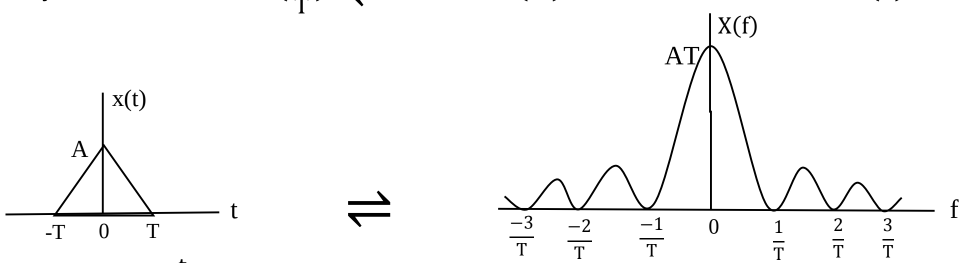
$$3\, \mathrm{rect}(\frac{t}{10}) \rightleftharpoons 30\, \mathrm{sinc}(10f)$$

, and $$6\, \mathrm{rect}(3t) \rightleftharpoons 2\, \mathrm{sinc}(\frac{f}{3})$$

2. **For The Triangle Pulse and its Sinc Squared Spectra:** Consider x(t) an energy signal, is a symmetric triangle pulse, centered at the origin, is expressed by x(t) = A tri(t/T), its frequency spectrum X(f), Fig.3, is given by

$$X(f) = \int_{-T}^{0} (A + \frac{A}{T} t)\, e^{-j2\pi ft}\, dt + \int_{0}^{T} (A - \frac{A}{T} t)\, e^{-j2\pi ft}\, dt = AT \frac{\sin^2(\pi fT)}{(\pi fT)^2} = AT\, \mathrm{sinc}^2(fT)$$

, or equivalently $$A\, \mathrm{tri}(\frac{t}{T}) \rightleftharpoons AT\, \mathrm{sinc}^2(fT) \qquad (7)$$




a) $x(t) = A\, \mathrm{tri}(\frac{t}{T})$ b) $X(f) = AT\, \mathrm{sinc}^2(fT)$

Fig.3. a) Triangle pulse, b) Sinc squared spectrum.

, such as the following Fourier transform pairs

$$5\, \mathrm{tri}(\frac{t}{2}) \rightleftharpoons 10\, \mathrm{sinc}^2(2f)$$

, and $$8\, \mathrm{tri}(4t) \rightleftharpoons 2\, \mathrm{sinc}^2(\frac{f}{4})$$

**3. For the Decaying and Rising Single Sided Exponential Pulses:** Consider the energy signals, are the decaying and rising single sided exponential pulses, Fig.4, are expressed by $x_1(t) = e^{-t}\,u(t)$, and $x_2(t) = e^{t}\,u(-t)$, respectively, where u(t) is the unit step function, their Fourier transforms $X_1(f)$ and $X_2(f)$, are given by

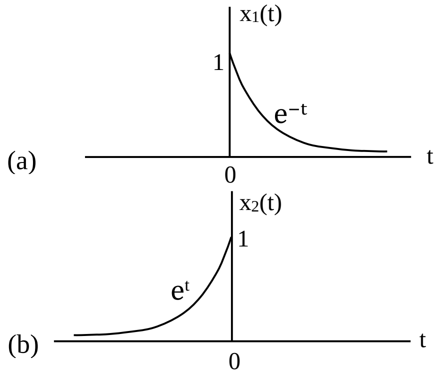


Fig.4. a) Single sided decaying signal,
b) Single sided rising signal.

$$X_1(f) = \int_0^{\infty} e^{-t}\, e^{-j2\pi ft}\, dt \; = \frac{1}{1 + j2\pi f}$$

, $$X_2(f) = \int_{-\infty}^{0} e^{t}\, e^{-j2\pi ft}\, dt \; = \frac{1}{1 - j2\pi f}$$

, and their Fourier transform pairs yield

$$e^{-t}\, u(t) \rightleftharpoons \frac{1}{1 + j2\pi f}$$

. $$e^{t}\, u(-t) \rightleftharpoons \frac{1}{1 - j2\pi f}$$

, or equivalently

$$e^{-t}\, u(t) \rightleftharpoons \frac{1}{\sqrt{1 + (2\pi f)^2}}\, e^{-j\tan^{-1}(2\pi f)} \qquad (8)$$

, $$e^{t}\, u(-t) \rightleftharpoons \frac{1}{\sqrt{1 + (2\pi f)^2}}\, e^{-j\tan^{-1}(-2\pi f)} \qquad (9)$$

, then the decaying and rising single sided exponential pulses $e^{-t}\,u(t)$ and $e^{t}\,u(-t)$, are mapped into complex functions in the frequency domain, where the amplitude spectrums are even functions and the phase spectrums are odd functions, Fig.5.

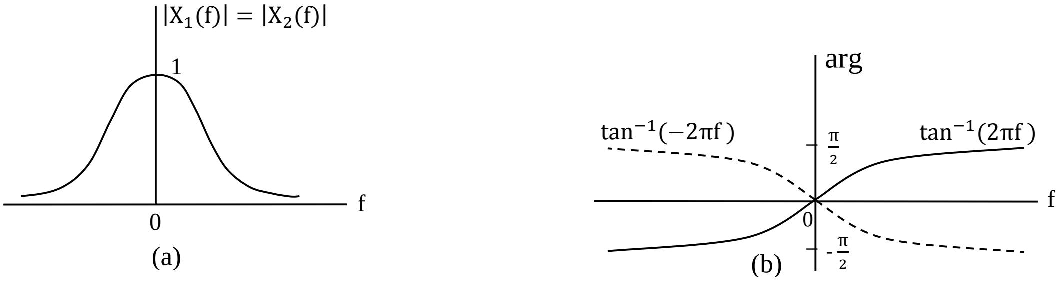


Fig.5. a) Magnitude spectrum of the single sided decaying and rising pulses.
b) Phase spectrum of the single sided decaying and rising pulses.

**Problem 1**

Prove that the inverse Fourier transform of the following complex spectrum

$$X(f) = \frac{1}{1 + j2\pi f}$$

, is the decaying single sided exponential pulse $x(t) = e^{-t}\,u(t)$, where u(t) is the unit step function.

*Solution*

The inverse Fourier transform x(t), Eq.(3), is given by

$$x(t) = \int_{-\infty}^{\infty} \frac{1}{1 + j2\pi f}\, e^{j2\pi ft}\, df$$

, let $w = 2\pi f$, then $dw = 2\pi\, df$, and as $f\to-\infty$ then $w\to-\infty$, and as $f\to\infty$ then $w\to\infty$, and the inverse Fourier transform x(t) yields

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1 + jw}\, e^{jwt}\, dw$$

, multiplying the numerator and denominator by $(1 - jw)$, x(t) yields

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1+jw)}\frac{(1-jw)}{(1-jw)} e^{jwt}\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{jwt}}{1+w^2}\, dw - \frac{1}{2\pi}\int_{-\infty}^{\infty} jw\,\frac{e^{jwt}}{1+w^2}\, dw \qquad (10)$$

, according to the following closed contour integration (Cauchy`s Residue theorem)

$$\int_c f(w)\, e^{jwt}\, dw = 2\pi j \sum_{k=1}^{\infty} \text{Residue}\,[\, f(w), e^{jwt}, k] \qquad \text{for} \quad t > 0$$

$$= 2\pi j \lim_{w\to k} (w - k)\, f(w)\, e^{jwt}$$

, where k is the number of poles inside the contour c which are poles at w = ± j, x(t), Eq.(10) yields

$$x(t) = \frac{1}{2\pi}\left[2\pi j \lim_{w\to j}(w - j)\, \frac{e^{jwt}}{(w-j)(w+j)}\right] - \frac{1}{2\pi}\left[2\pi j \lim_{w\to j}(w - j)\, \frac{jw\, e^{jwt}}{(w-j)(w+j)}\right]$$

$$= \frac{1}{2\pi}\left[2\pi j\, \frac{e^{-t}}{2j}\right] - \frac{1}{2\pi}\left[2\pi j\, \frac{(-e^{-t})}{2j}\right] \qquad \text{for} \quad t > 0$$

, or equivalently

$$x(t) = e^{-t}\, u(t)$$

, then the following integral will be valued

$$\int_{-\infty}^{\infty} \frac{1}{1+j2\pi f}\, e^{j2\pi ft}\, df = e^{-t}\, u(t)$$

, also using the same procedures for the rising single sided exponential pulse, Eq.(9).

### 3. For The Linearity Property:

Let $x_1(t) \rightleftharpoons X_1(f)$ , and $x_2(t) \rightleftharpoons X_2(f)$

The Fourier transform of the sum of two energy signals, is the sum of the Fourier transform of the two signals, where

$$a_1\, x_1(t) + a_2\, x_2(t) \rightleftharpoons a_1\, X_1(f) + a_2\, X_2(f)$$

, where $a_1$ and $a_2$ are constants.

**Problem 2**

Find the Fourier transform of the energy signals, $x_1(t)$ and $x_2(t)$, Fig.6, using linearity property.

*Solution*

The signals $x_1(t)$ and $x_2(t)$ are the algebraic summation of a rectangular and a triangle signals, and may be expressed by

$$x_1(t) = 5\,\text{rect}(\frac{t}{6}) + 3\,\text{tri}(\frac{t}{3})$$

, and $x_2(t) = 5\,\text{rect}(\frac{t}{6}) - 3\,\text{tri}(\frac{t}{3})$

, due to the linearity property, and the Fourier transforms of the rectangular and triangle pulses, Equations (6) and (7), their Fourier transforms $X_1(f)$ and $X_2(f)$ yield

$$X_1(f) = 30\,\text{sinc}(6f) + 9\,\text{sinc}^2(3f) \qquad (11)$$

, and $X_2(f) = 30\,\text{sinc}(6f) - 9\,\text{sinc}^2(3f) \qquad (12)$

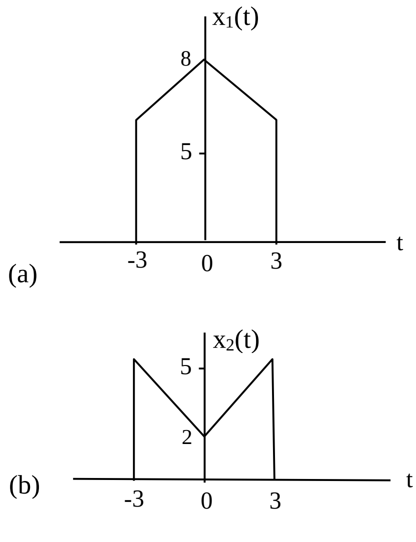



Fig.6. $x_1(t)$ and $x_2(t)$ are the algebraic summation of rectangular and triangle pulses.

, these Fourier transforms can also be obtained, using Fourier integrals, Eq.(3), $X_1(f)$ and $X_2(f)$ yield

$$X_1(f) = \int_{-3}^{0} (8+t)\, e^{-j2\pi ft}\, dt + \int_{0}^{3} (8-t)\, e^{-j2\pi ft}\, dt$$

, and $$X_2(f) = \int_{-3}^{0} (2-t)\, e^{-j2\pi ft}\, dt + \int_{0}^{3} (2+t)\, e^{-j2\pi ft}\, dt$$

, using the integration by parts technique, express the exponential terms by sine functions, the same spectrums, Equations (11) and (12), will be obtained, and the Fourier transform pairs will be

$$5\,\text{rect}(\frac{t}{6}) + 3\,\text{tri}(\frac{t}{3}) \rightleftharpoons 30\,\text{sinc}(6f) + 9\,\text{sinc}^2(3f)$$

, and $$5\,\text{rect}(\frac{t}{6}) - 3\,\text{tri}(\frac{t}{3}) \rightleftharpoons 30\,\text{sinc}(6f) - 9\,\text{sinc}^2(3f)$$

**Problem 3**

Find the Fourier transform of the double sided exponential pulse x(t), Fig.7a, using linearity property, where x(t) is given by

$$x(t) = e^{-|t|}$$

*Solution*

The pulse $x(t) = e^{-|t|}$, is the algebraic summation of the decaying and rising single sided exponential pulses $e^{-t}\,u(t)$ and $e^{t}\,u(-t)$, Fig.4, then x(t) is given by

$$e^{-|t|} = e^{-t}\,u(t) + e^{t}\,u(-t)$$

, due to the linearity property, an the Fourier transforms of the decaying and rising single sided exponential pulses, Equations (8) and (9), the Fourier transform X(f), Fig.7b, yields

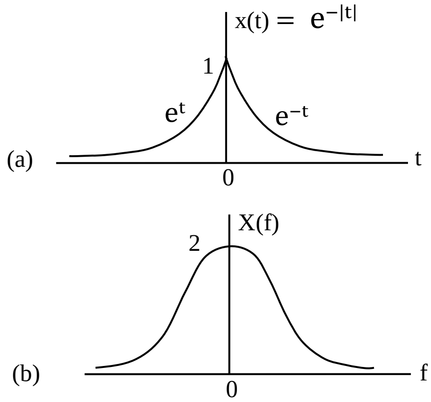



Fig.7. a) Double sided exponential pulse.
b) The Fourier transform of x(t) and has zero phase angle.

$$X(f) = \frac{1}{1+j2\pi f} + \frac{1}{1-j2\pi f} = \frac{2}{1+(2\pi f)^2} \quad (13)$$

, or equivalently $$e^{-|t|} \rightleftharpoons \frac{2}{1+(2\pi f)^2} \quad (14)$$

**Problem 4**

Given two energy signals $x_1(t)$ and $x_2(t)$, Fig.8, are given by

$$x_1(t) = 2\,\text{rect}(\frac{t}{8}) \quad ,\text{and} \quad x_2(t) = 3\,\text{tri}(\frac{t}{6})$$

Find the Fourier transform of the energy signal x(t), where x(t) is the product of $x_1(t)$ by $x_2(t)$, using the linearity property.

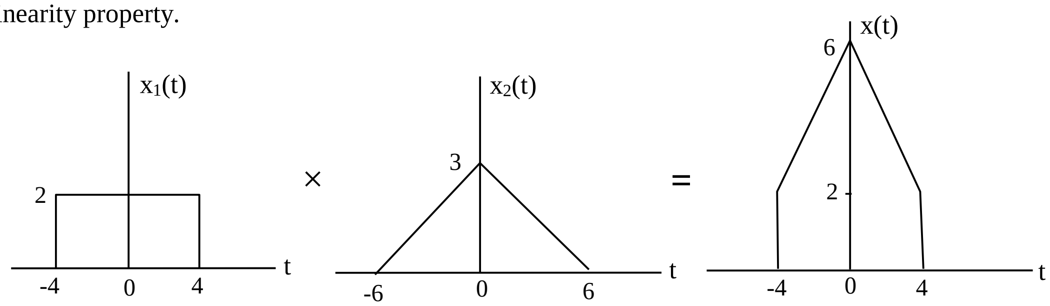



Fig.8. The graphical multiplication of rectangular and triangle pulses.

*Solution*

Since the signal x(t) is the product of $x_1(t)$ and $x_2(t)$, x(t) may expressed by

$$x(t) = 2\,\text{rect}(\frac{t}{8}) \times 3\,\text{tri}(\frac{t}{6})$$

, Fig.8. shows the graphical multiplication of $x_1(t)$ by $x_2(t)$, and the signal x(t) is expressed by

$$x(t) = 2\,\text{rect}(\frac{t}{8}) + 4\,\text{tri}(\frac{t}{4})$$

, due to the Fourier transform of the rectangular and triangle pulses, Equations (6) and (7). The Fourier transform X(f) will be

$$X(f) = 16\,\text{sinc}(8f) + 16\,\text{sinc}^2(4f) \qquad (15)$$

, this Fourier transform can also be obtained, using Fourier integral, Eq.(3), X(f) yields

$$X(f) = \int_{-4}^{0} (6+t)\, e^{-j2\pi ft}\, dt + \int_{0}^{4} (6-t)\, e^{-j2\pi ft}\, dt$$

, using the integration by parts technique, express the exponential terms by sine functions, the same spectrum, Eq.(15), will be obtained, and the Fourier transform pair will be

$$2\,\text{rect}(\frac{t}{8}) \times 3\,\text{tri}(\frac{t}{6}) \rightleftharpoons 16\,\text{sinc}(8f) + 16\,\text{sinc}^2(4f)$$

**Problem 5**

Find the Fourier transform of the energy signals, $x_1(t)$ and $x_2(t)$, Fig.9, hence, evaluate the Fourier transform of the function $x(t) = x_1(t) + x_2(t)$, using linearity property.

*Solution*

The Fourier transforms of the signals, $x_1(t)$ and $x_2(t)$, Eq.(2), are given by

$$X_1(f) = \int_{0}^{2} 7\, e^{-j2\pi ft}\, dt = 14\,\text{sinc}(2f)\, e^{-j2\pi f}$$

$$, \quad X_2(f) = \int_{-2}^{0} 7\, e^{-j2\pi ft}\, dt = 14\,\text{sinc}(2f)\, e^{j2\pi f}$$

, due to linearity property, the Fourier transform X(f) of the signal $x(t) = x_1(t) + x_2(t)$, will be

$$X(f) = 14\,\text{sinc}(2f)\, e^{-j2\pi f} + 14\,\text{sinc}(2f)\, e^{j2\pi f}$$

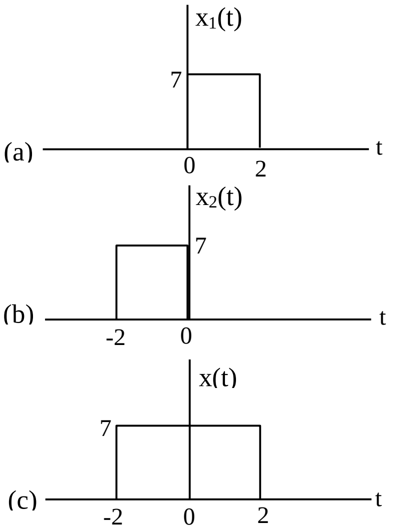



Fig.9. The summation of two shifted pulses, a) $x_1(t)$, b) $x_2(t)$ c) $x(t) = x_1(t) + x_2(t)$.

, express the exponential terms by cosine function and using the formula " sin(2x) = 2 sin(x) cos(x)", and in terms of sinc function, X(f) yields $X(f) = 28\,\text{sinc}(4f)$ , which is the same Fourier transform of the signal x(t), Fig.9c, is given by $x(t) = 7\,\text{rect}(t/4)$.

**Problem 6**

Find the Fourier transform of the energy signals, x(t), Fig.10a, using linearity property.

*Solution*

The energy signal x(t), Fig.10a, is the summation of the energy signals $x_1(t)$ and $x_2(t)$, Fig.10b,c , and using linearity property, can be written in the form

$$x_1(t) = \left[12\,\text{tri}(\frac{t}{4}) - 9\,\text{tri}(\frac{t}{3}) - 3\,\text{rect}(\frac{t}{6})\right]$$

$$, x_2(t) = \left[6\,\text{rect}(\frac{t}{6}) - 10\,\text{tri}(t)\right]$$

, and their Fourier transform are given by

$$X_1(f) = 48\,\text{sinc}^2(4f) - 27\,\text{sinc}^2(3f) - 18\,\text{sinc}(6f)$$

$$, X_2(f) = 36\,\text{sinc}(6f) - 10\,\text{sinc}^2(f)$$

, since the energy signal x(t) is the summation of the signals $x_1(t)$ and $x_2(t)$, x(t) can be written in the form

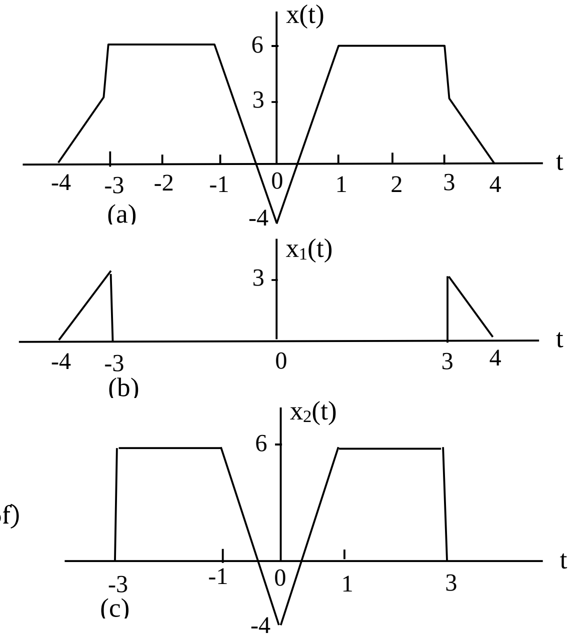



Fig.9. The summation of two energy signals, a) $x(t) = x_1(t) + x_2(t)$, b) $x_1(t)$, c) $x_2(t)$.

$$x(t) = \left[12\,\text{tri}(\frac{t}{4}) - 9\,\text{tri}(\frac{t}{3}) - 3\,\text{rect}(\frac{t}{6})\right] + \left[6\,\text{rect}(\frac{t}{6}) - 10\,\text{tri}(t)\right]$$

, and its Fourier transform is given by

$$X(f) = 48\,\mathrm{sinc}^2(4f) - 27\,\mathrm{sinc}^2(3f) - 18\,\mathrm{sinc}(6f) + 36\,\mathrm{sinc}(6f) - 10\,\mathrm{sinc}^2(f)$$

, or equivalently

$$X(f) = 18\,\mathrm{sinc}(6f) + 48\,\mathrm{sinc}^2(4f) - 27\,\mathrm{sinc}^2(3f) - 10\,\mathrm{sinc}^2(f)$$

**Problem 7**

Find the Fourier transform of the energy signals, $x_1(t)$ and $x_2(t)$, Fig.6a,b, and then find the Fourier transform of the energy signal x(t), Fig.6c, where $x(t) = x_1(t) + x_2(t)$, using linearity property.

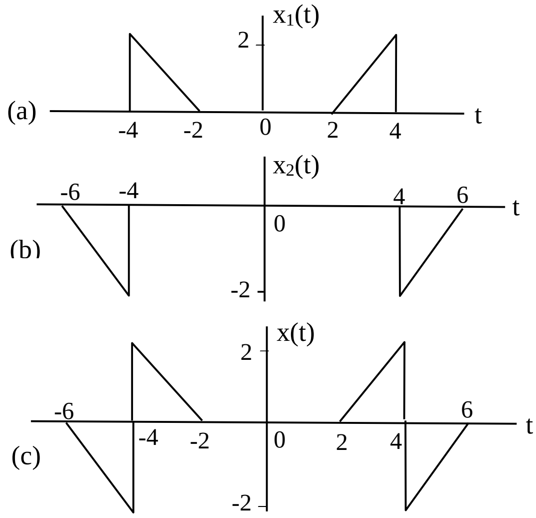



Fig.9. The summation of two shifted pulses,
a) $x_1(t)$, b) $x_2(t)$ c) $x(t) = x_1(t) + x_2(t)$.

*Solution*

Using the linearity property, $x_1(t)$ and $x_2(t)$ can be written in the form

$$x_1(t) = 2\,\mathrm{rect}\left(\frac{t}{8}\right) - 4\,\mathrm{tri}\left(\frac{t}{4}\right) + 2\,\mathrm{tri}\left(\frac{t}{2}\right)$$

$$, x_2(t) = -6\,\mathrm{tri}\left(\frac{t}{6}\right) + 2\,\mathrm{rect}\left(\frac{t}{8}\right) + 4\,\mathrm{tri}\left(\frac{t}{4}\right)$$

, then their Fourier transform are given by

$$X_1(f) = 16\,\mathrm{sinc}(8f) - 16\,\mathrm{sinc}^2(4f) + 4\,\mathrm{sinc}^2(2f)$$

$$, X_2(f) = -36\,\mathrm{sinc}^2(6f) + 16\,\mathrm{sinc}(8f) + 16\,\mathrm{sinc}^2(4f)$$

, then the energy signal x(t) is given by

$$x(t) = 2\,\mathrm{rect}\left(\frac{t}{8}\right) - 4\,\mathrm{tri}\left(\frac{t}{4}\right) + 2\,\mathrm{tri}\left(\frac{t}{2}\right) - 6\,\mathrm{tri}\left(\frac{t}{6}\right) + 2\,\mathrm{rect}\left(\frac{t}{8}\right) + 4\,\mathrm{tri}\left(\frac{t}{4}\right)$$

, and its Fourier transform is given by

$$X(f) = 16\,\mathrm{sinc}(8f) - 16\,\mathrm{sinc}^2(4f) + 4\,\mathrm{sinc}^2(2f) - 36\,\mathrm{sinc}^2(6f) + 16\,\mathrm{sinc}(8f) + 16\,\mathrm{sinc}^2(4f)$$

, or equivalently

$$X(f) = 32\,\mathrm{sinc}(8f) - 16\,\mathrm{sinc}^2(4f) + 4\,\mathrm{sinc}^2(2f) - 36\,\mathrm{sinc}^2(6f)$$

**4. Conclusion**: The efficient utilization of the Fourier transform properties leads to time saving of the numerical computation algorithms for the design of the communication and signal processing problems which need long time solutions. Linearity property facilitate the solution of these problems and may be evaluate the Fourier transform almost by inspection. It is necessary to study how this property allow the easy solution of these problems with efficient utilization otherwise will be quite difficult.

**References**

Khamies El-Shennawy, Author, book title, “Communication Theory and Signal Processing for Transform Coding”, Bentham Science Publishers, 2014, eISBN: 978-1-60805-830-3, and ISBN: 978-1-60805-831-0.

## Biography

Dr. Khamies El-Shennawy is a Prof. of Marine Communications in The Arab Academy of Science & Technology and Maritime Transport AASTMT, Alexandria, Egypt. He was the President Assistant of the Academy for technology transfer. His field of interests: distributed networks, Charge Coupled Devices CCD and Surface Acoustic Wave SAW devices for mobile communication and Bio-Sensors, data communications, speech coding, speech enhancement, Voice over Internet Protocol VoIP, communication security systems, speech and image watermarking, audio and video compression, acoustics, Ultra Wide Band UWB wireless communication systems, Multi Carrier Direct Sequence Code Division Multiple Access MC-DS-CDMA, Worldwide interoperability for Microwave Access WiMAX and Vehicular AdHoc Networks VANETS, Electronic Chart Display Information Systems ECDIS, Global Positioning Systems GPS, Global Maritime Distress and Safety Systems GMDSS, air-borne and space-borne remote sensing.