# Numerical Computation of Discrete Fourier Transform Circular Convolution Property

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**Abstract:** Numerical computation of Fourier transform is particularly well suited for use on a Digital computer, and is defined by Discrete Fourier Transform DFT and Fast Fourier Transform FFT. Also, more efficient transforms such as Discrete Cosine Transform DCT, Discrete Wavelet Transform DWT, and Contourlet Transform CT are developed. Discrete Fourier Transform DFT requires a discrete input function whose non-zero values have limited (finite) duration. Therefore it is often said that the DFT is a transform for the Fourier analysis of finite domain, such inputs are often created by sampling the continuous function (sampling theorem). Since the input function is a finite sequence of real or complex numbers, the DFT is ideal for processing information stored in computers. The DFT is widely employed in signal processing and related fields to analyze the frequencies contained in a sampled signal, and to solve the partial differential equations and also to perform other operations such as convolution processes.

## 1. Discrete Fourier Transform

Since the Discrete Fourier Transform DFT requires a discrete input function. Let x(t) is a continuous signal with finite duration, and the uniform sampling of x(t) is done to obtain a finite sequence of samples $x(nT_s)$, with sampling period $T_s$ , and $1/T_s$ is the sampling rate must equal to or greater than twice the highest frequency component of the signal x(t). If N is the number of samples, then $x(nT_s)$ is the data array sequence, denoted by: $x(0)$, $x(T_s)$, $x(2T_s)$, … , $x[(N-1)T_s]$, where the index n takes the values from 0 to $(N-1)$.

The discrete Fourier transform of the data array sequence $x(nT_s)$ may be defined by $X(kf_s)$, and can be obtained by using the trapezoidal rule, Fig.1, for approximating the integral which defines the Fourier transform of the given function x(t), where the dashed area equals " $T_s\,x(nT_s)$ ". The Fourier transform $X(kf_s)$ consists of another data array sequence of samples N separated in the frequency by $f_s$ Hz, is given by

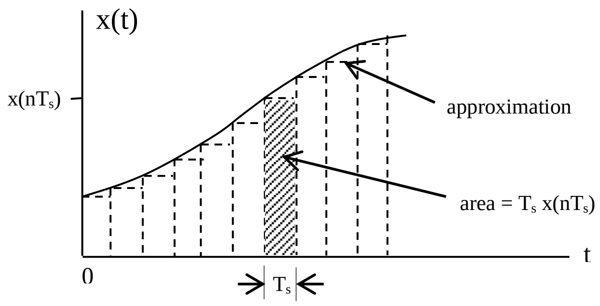


Fig.1. An approximated $x(nT_s)$.

$$X(kf_s) = \sum_{n=0}^{N-1} T_s\, x(nT_s)\, e^{-j2\pi\, nT_s kf_s} \qquad , \quad k = 0,1,2,3,\dots,(N-1) \qquad (1)$$

, since N denote the number of frequency samples contained in an interval $f_s$ . Hence, the frequency resolution involved in the numerical computation of the Fourier transform is defined by

$$\Delta f = \frac{f_s}{N} = \frac{1}{NT_s} = \frac{1}{T_N}$$

, where $T_N$ is the total duration of the signal x(t), then the product of the parameters $T_s$ and $f_s$ , are related by $T_s f_s = 1/N$, and $X(kf_s)$ yields

$$X(kf_s) = T_s \sum_{n=0}^{N-1} x(nT_s)\, e^{-j2\pi \frac{nk}{N}} \qquad , \quad k = 0,1,2,3,\dots,(N-1) \qquad (2)$$

, the inverse discrete Fourier transform $x(nT_s)$ of Eq.(2) can be obtained by multiplying both sides of Eq.(2) by $e^{j2\pi\, mk/N}$, and sum over the index k from 0 to $(N-1)$, yields

$$\sum_{k=0}^{N-1} X(kf_s)\, e^{j2\pi\, mk/N} = T_s \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} x(nT_s)\, e^{j2\pi \frac{k(m-n)}{N}} \qquad (3)$$

, interchange the order of the summation, and using the fact that

$$\sum_{k=0}^{N-1} e^{j2\pi \frac{k(m-n)}{N}} = N \qquad \text{for } m = n$$
$$= 0 \qquad \text{otherwise}$$

, Eq.(3) yields

$$x(nT_s) = \frac{1}{NT_s} \sum_{k=0}^{N-1} X(kf_s)\, e^{j2\pi \frac{nk}{N}} \qquad , n = 0, 1, 2, 3, \dots, (N-1) \qquad (4)$$

, $x(nT_s)$ and $X(kf_s)$ constitute a discrete Fourier transform pair and each of them is called the mate of the other, and each of them can be obtained in terms of the other, Equations (2) and (4). The symbol $X(kf_s) = DFT[x(nT_s)]$ indicates the discrete Fourier transform operation, Eq.(2), and the symbol $x(nT_s) = DFT^{-1}[X(kf_s)]$ indicates the inverse discrete Fourier transform operation, Eq.(4). Another most convenient symbol more frequently used, indicates the discrete Fourier transform pair, is given by

$$x(nT_s) \rightleftharpoons X(kf_s)$$

An important feature of the DFT is that, the signal $x(nT_s)$ and its spectrum $X(kf_s)$ are both in discrete form, and since the DFT is a transform for Fourier analysis of finite domain, and requires a discrete input function whose non-zero values have limited (finite) duration. Then $x(nT_s)$ and $X(kf_s)$ are periodic with the period of either one, consisting of a finite number of samples N, that is

$$x(nT_s) = x(nT_s + NT_s)$$

, and also its spectrum

$$X(kf_s) = X(kf_s + Nf_s)$$

, this discrete Fourier transform pair is an exact and faithful relationship. There is no error in going from the data sequence $x(nT_s)$ to the data sequence $X(kf_s)$ and vice versa. The sequence $X(kf_s)$ is periodic of period N, as are previously explained and as one can easily verify. The process of sampling the period N whether or not, the original sequence $x(nT_s)$ is periodic. This easily verified by calculating $x(nT_s + NT_s)$, as follow from Eq.(4)

$$x(nT_s + NT_s) = \frac{1}{NT_s} \sum_{k=0}^{N-1} X(kf_s)\, e^{j2\pi \frac{(n+N)k}{N}}$$
$$= \frac{1}{NT_s} \sum_{k=0}^{N-1} X(kf_s)\, e^{j2\pi \frac{nk}{N}}\, e^{j2\pi k} \qquad (5)$$

, since the value of the exponent $e^{j2\pi k}$ equals unity, where $k = 0, 1, 2, 3, \dots, (N-1)$, Eq.(5) yields

$$x(nT_s + NT_s) = \frac{1}{NT_s} \sum_{k=0}^{N-1} X(kf_s)\, e^{j2\pi \frac{nk}{N}}$$

, or equivalently

$$x(nT_s + NT_s) = x(nT_s) \qquad , n = 0, 1, 2, 3, \dots, (N-1) \qquad (6)$$

, N is the number of the samples of the data sequence, in other words, the sampling in one domain induces periodicity in the other domain. When the samples in a time series, its spectrum becomes periodic. On the other hand, when sampling a spectrum of a time series, the time series is periodically extended. The discrete Fourier transform $X(kf_s)$, Eq.(2), and the inverse discrete Fourier transform $x(nT_s)$, Eq.(4), are complex quantities, they contain the real cosine terms and the imaginary sine terms, so they may be have even magnitude discrete spectrum and odd phase discrete spectrum [1].

Considering the sampling period $T_s$ equals unity, Equations (2) and (4), yield

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi \frac{nk}{N}} \qquad , k = 0, 1, 2, 3, \ldots , (N-1) \qquad (7)$$

, and

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi \frac{nk}{N}} \qquad , n = 0, 1, 2, 3, \ldots , (N-1) \qquad (8)$$

, Equations (7) and (8) illustrate that any numerical computation algorithm for computing X(k) from the sequence x(n), can also be used to compute x(n) from the sequence X(k), where it is necessary only to multiply by (1/N) and change the sign of the exponent in the summation.

**Problem 1**

Evaluate the discrete Fourier transform of the following data array sequence "1 , 1 , 0 , 0, 0, 0", consider the sampling period $T_s$ equals unity.

*Solution*

The number of samples N equals 6, and Eq.(7) yields

$$X(k) = \sum_{n=0}^{5} x(n)\, e^{-j\pi nk/3} \qquad , \quad k = 0, 1, 2, 3, 4, \text{and } 5.$$

, then the discrete frequency domain X(0), X(1), X(2), X(3), X(4), and X(5) are given by

$$X(0) = \sum_{n=0}^{5} x(n) \qquad = 1 + 1 + 0 + 0 + 0 + 0 = 2$$

,

$$X(1) = \sum_{n=0}^{5} x(n)\, e^{-j\pi n/3} = 1 + e^{-j\pi/3} + 0 + + 0 + 0 = \frac{3}{2} - j\,\frac{\sqrt{3}}{2}$$

,

$$X(2) = \sum_{n=0}^{5} x(n)\, e^{-j\pi 2n/3} = 1 + e^{-j2\pi/3} + 0 + 0 + 0 + 0 = \frac{1}{2} - j\,\frac{\sqrt{3}}{2}$$

,

$$X(3) = \sum_{n=0}^{5} x(n)\, e^{-j\pi n} = 1 + e^{-j\pi} + 0 + 0 + 0 + 0 = 0$$

,

$$X(4) = \sum_{n=0}^{5} x(n)\, e^{-j\pi 4n/3} = 1 + e^{-j4\pi/3} + 0 + 0 + 0 + 0 = \frac{1}{2} + j\,\frac{\sqrt{3}}{2}$$

, and

$$X(5) = \sum_{n=0}^{5} x(n)\, e^{-j\pi 5n/3} = 1 + e^{-j5\pi/3} + 0 + 0 + 0 + 0 = \frac{3}{2} + j\,\frac{\sqrt{3}}{2}$$

, then the discrete Fourier transform of the data array sequence "1 , 1 , 0 , 0 , 0 , 0", is another data array sequence "2 , $(\frac{3}{2} - j\,\frac{\sqrt{3}}{2})$ , $(\frac{1}{2} - j\,\frac{\sqrt{3}}{2})$ , 0 , $(\frac{1}{2} + j\,\frac{\sqrt{3}}{2})$ , $(\frac{3}{2} + j\,\frac{\sqrt{3}}{2})$".

**Problem 2**

Evaluate the inverse discrete Fourier transform of the following data array sequence "1 , 0 , 0 , 0, 1, 0, 0, 0", consider the sampling period $T_s$ equals unity.

*Solution*

The number of samples N equals 8, and Eq.(8) yields

$$x(n) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{j\pi nk/4} \qquad , \quad n = 0, 1, 2, 3, 4, 5, 6, \text{and } 7.$$

, then the discrete time domain x(0), x(1), x(2), x(3), x(4), x(5), x(6), and x(7) are given by

$$x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k) \qquad = \frac{1}{8} \, [\, 1+0+0+0+1+0+0+0 \,] = \frac{1}{4}$$

$$,\ x(1) = \frac{1}{8} \sum_{k=0}^{7} X(k)\ e^{j\pi\, k/4} = \frac{1}{8} \left[ 1+0+0+0+\ e^{j\pi}\ +0+0+0 \right] = 0$$

$$,\ x(2) = \frac{1}{8} \sum_{k=0}^{7} X(k)\ e^{j\pi\, k/2} = \frac{1}{8} \left[ 1+0+0+0+\ e^{j2\pi} +0+0+0 \right] = \frac{1}{4}$$

$$,\ x(3) = \frac{1}{8} \sum_{k=0}^{7} X(k)\ e^{j3\pi\, k/4} = \frac{1}{8} \left[ 1+0+0+0+\ e^{j3\pi} +0+0+0 \right] = 0$$

$$,\ x(4) = \frac{1}{8} \sum_{k=0}^{7} X(k)\ e^{j\pi k} \qquad = \frac{1}{8} \left[ 1+0+0+0+\ e^{j4\pi} +0+0+0 \right] = \frac{1}{4}$$

$$,\ x(5) = \frac{1}{8} \sum_{k=0}^{7} X(k)\ e^{j\pi\, 5k/4} = \frac{1}{8} \left[ 1+0+0+0+\ e^{j5\pi} +0+0+0 \right] = 0$$

$$,\ x(6) = \frac{1}{8} \sum_{k=0}^{7} X(k)\ e^{j\pi\, 3k/2} = \frac{1}{8} \left[ 1+0+0+0+\ e^{j6\pi} +0+0+0 \right] = \frac{1}{4}$$

, and

$$x(7) = \frac{1}{8} \sum_{k=0}^{7} X(k)\ e^{j\pi\, 7k/4} = \frac{1}{8} \left[ 1+0+0+0+\ e^{j7\pi} +0+0+0 \right] = 0$$

, then the inverse discrete Fourier transform of the data array sequence “1 , 0 , 0 , 0 , 1 , 0 , 0 , 0”, is another data array sequence "$\frac{1}{4}$ , 0 , $\frac{1}{4}$ , 0 , $\frac{1}{4}$ , 0 , $\frac{1}{4}$ , 0".

**2. DFT Convolution Property (Circular Convolution)**

Many Fourier transform FT problems need long time solutions, so FT properties facilitate the solution of these problems and may be evaluated almost by inspection [2]. It is necessary to study how these properties allow the easy solution of these problems with efficient utilization. The properties of the DFT are similar to those of the continuous Fourier transform, except the periodicity of the data array sequences, however, introduces certain basic differences [1].

Let the following discrete Fourier transform pairs

$$x_1(nT_s) \rightleftharpoons X_1(kf_s)$$

, and

$$x_2(nT_s) \rightleftharpoons X_2(kf_s)$$

, and if N is the number of data array sequence, n = 0, 1, 2, … , (N − 1), and k = 0, 1, 2, … , (N − 1), the convolution property discrete Fourier transform pair is

$$x_1(nT_s) \otimes x_2(nT_s) \rightleftharpoons X_1(kf_s)\, X_2(kf_s)$$

, where the symbol ⊗ denotes the convolution process, and the shorthand notation $x_1(nT_s) \otimes x_2(nT_s)$ is frequently used and called the convolution of $x_1(nT_s)$ and $x_2(nT_s)$. In the discrete time domain, the convolution signal $x_{12}(nT_s)$ is given by

$$x_{12}(nT_s) = x_1(nT_s) \otimes x_2(nT_s)$$

$$= T_s \sum_{i=0}^{2N-1} x_1(iT_s)\ x_2[(n-i)T_s] \qquad ,\ n = 0, 1, 2, \dots, (2N-1) \qquad (9)$$

, the indices i and n of the convolution function $x_{12}(nT_s)$, Eq.(9), ly between zero and (2N − 1) because since the bandwidth sequence of $x_1(nT_s)$ and $x_2(nT_s)$ is N samples, n = 0, 1, 2, …, (N – 1), then the bandwidth sequence of $x_{12}(nT_s)$ is (2N – 1), where n = 0, 1, 2, … , (2N − 1).

But if the bandwidth sequences of $x_1(nT_s)$ and $x_2(nT_s)$ are $N_1$ and $N_2$ respectively. Then the bandwidth sequence of the convoluted signal $x_{12}(nT_s)$ will be $N_c = N_1 + N_2 - 1$, where n = 0, 1, 2, … , $(N_1 + N_2 - 1)$.

For the evaluation of the circular convolution function $x_{12}(nT_s) = x_1(nT_s) \otimes x_2(nT_s)$, since $x_1(nT_s)$ and $x_2(nT_s)$ are periodic in their $N_1$ and $N_2$ data sequences respectively, in this case, the circular convolution function $x_{12}(nT_s)$ of the successive periods can overlap in the convolution sum, so some additional thought must be taken into consideration by adding a guard band of zeroes at the ends of their sequences so that successive periods can not overlap in the convolution sum. The length of the guard band depends on the data sequence N of $x(nT_s)$ and equals (N − 1), in this case the convolution function $x_{12}(nT_s)$ can be obtained and repeated periodically, Fig.2, Example 3.

**Example 3**

Evaluate the circular convolution signal $x_{12}(nT_s) = x_1(nT_s) \otimes x_2(nT_s)$, where $x_1(nT_s)$ and $x_2(nT_s)$ are given by the following data array sequences: $x_1(nT_s)$ = 1, 3, 2, 1, and $x_2(nT_s)$ = 1, 2, 3, 4, and consider the sampling period $T_s$ equals unity.

*Solution*

The bandwidth sequences of $x_1(n)$ and $x_2(n)$ are $N_1$ equals 4 and $N_2$ also equals 4, where n = 0, 1, 2, and 3. To avoid the overlap of the successive periods in the convolution sum, a guard band of three zeroes (N – 1) at the ends of the sequences $x_1(n)$ and $x_2(n)$ are added, Fig.2a,b. The length of the guard band is three zeroes for both $x_1(n)$ and $x_2(n)$. The bandwidth sequence of the circular convoluted signal $x_{12}(n)$ will be $N_c = N_1 + N_2 - 1 = 7$, where n = 0, 1, 2, 3, 4, 5, 6. The convolution function $x_{12}(n)$, Fig.2c. can be obtained and repeated periodically, then using Fig.2a,b, Eq.(9) yields

$$x_{12}(n) = \sum_{i=0}^{6} x_1(i)\ x_2(n-i) \qquad , \quad n = 0, 1, 2, 3, 4, 5 \text{ and } 6$$

, then the discrete time domain $x_{12}(0)$, $x_{12}(1)$, $x_{12}(2)$, $x_{12}(3)$, $x_{12}(4)$, $x_{12}(5)$, and $x_{12}(6)$, are given by

$$x_{12}(0) = \sum_{i=0}^{6} x_1(i)\ x_2(-i) = 1\times1 + 3\times0 + 2\times0 + 1\times0 + 0\times4 + 0\times3 + 0\times2 = 1$$

$$x_{12}(1) = \sum_{i=0}^{6} x_1(i)\ x_2(1-i) = 1\times2 + 3\times1 + 2\times0 + 1\times0 + 0\times0 + 0\times4 + 0\times3 = 5$$

$$x_{12}(2) = \sum_{i=0}^{6} x_1(i)\ x_2(2-i) = 1\times3 + 3\times2 + 2\times1 + 1\times0 + 0\times0 + 0\times0 + 0\times4 = 11$$

$$x_{12}(3) = \sum_{i=0}^{6} x_1(i)\ x_2(3-i) = 1\times4 + 3\times3 + 2\times2 + 1\times1 + 0\times1 + 0\times2 + 0\times0 = 18$$

$$x_{12}(4) = \sum_{i=0}^{6} x_1(i)\ x_2(4-i) = 1\times0 + 3\times4 + 2\times3 + 1\times2 + 0\times1 + 0\times0 + 0\times0 = 20$$

$$x_{12}(5) = \sum_{i=0}^{6} x_1(i)\ x_2(5-i) = 1\times0 + 3\times0 + 2\times4 + 1\times3 + 0\times2 + 0\times1 + 0\times0 = 11$$

$$x_{12}(6) = \sum_{i=0}^{6} x_1(i)\ x_2(6-i) = 1\times0 + 3\times0 + 2\times0 + 1\times4 + 0\times3 + 0\times2 + 0\times1 = 4$$

, then the circular discrete convolution signal $x_{12}(n)$, Fig.2c, is given by

$$x_{12}(n) = 1, 3, 2, 1 \otimes 1, 2, 3, 4 = 1, 5, 11, 18, 20, 11, 4$$

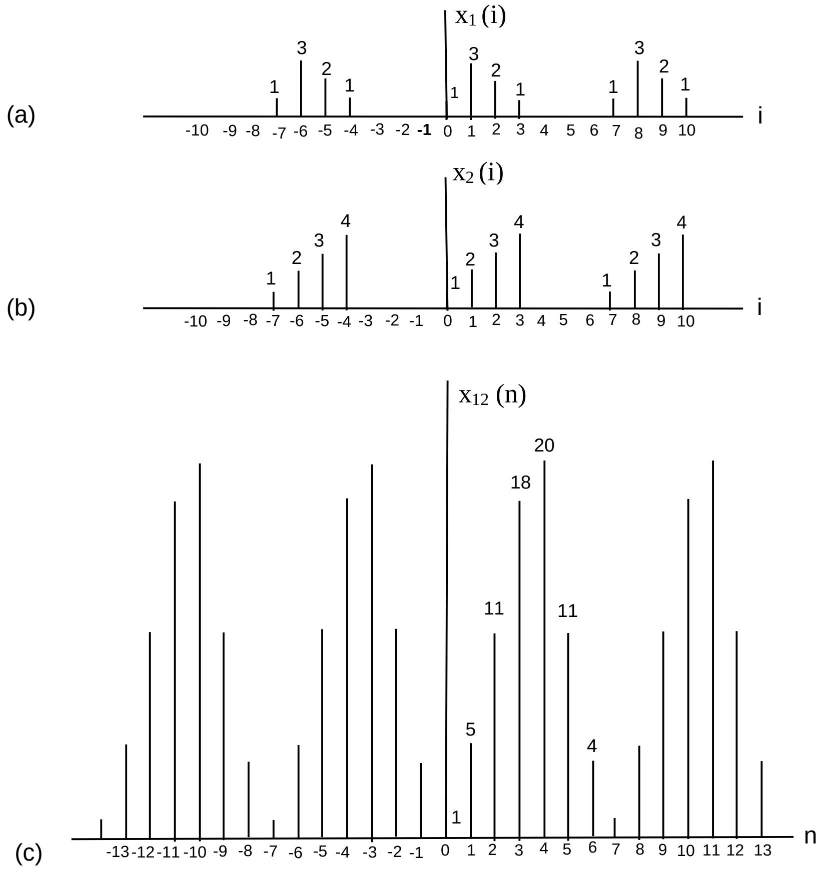


Fig.2. a) $x_1(i)$ with guard band, b) $x_2(i)$ with guard band, c) The circular convolution $x_{12}(n)$.

**4. Conclusion**: DFT Convolution Property (Circular Convolution algorithm) is an efficient utilization in the numerical computation of the Discrete Fourier Transform DFT and Fast Fourier Transform FFT on the digital computer. DFT properties are similar to those of the continuous Fourier transform, except the periodicity of the data array sequences, however, introduces certain basic differences, facilitates the solution of the communications problems and may evaluate the Fourier transform almost by inspection.

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## Biography

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